



TITLE:

Shape Optimization in Multi-Phase Stefan Problem(Evolution Equations and Applications to Nonlinear Problems)

AUTHOR(S):

KADOYA, Atsushi

CITATION:

KADOYA, Atsushi. Shape Optimization in Multi-Phase Stefan Problem(Evolution Equations and Applications to Nonlinear Problems). 数理解析研究所講究録 1991, 755: 185-200

ISSUE DATE:

1991-06

URL:

<http://hdl.handle.net/2433/82124>

RIGHT:

Shape Optimization in Multi-Phase Stefan Problem

Atsushi KADOYA (角谷 敦)

Department of Mathematics
Graduate School of Science and Technology
Chiba University

1. Formulation of the optimization problem

Let us consider the enthalpy formulation of Stefan problem described as follows:

$$SP(\Omega) \begin{cases} u_t - \Delta \beta(u) = f & \text{in } Q(\Omega) := (0, T) \times \Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ \beta(u) = g & \text{on } \Sigma(\Omega) := (0, T) \times \partial\Omega, \end{cases}$$

where $\hat{\Omega}$ is a fixed smooth bounded domain in R^N ($N \geq 2$), and Ω is a smooth subdomain of $\hat{\Omega}$, $0 < T < \infty$, $\hat{Q} := (0, T) \times \hat{\Omega}$ and $\hat{\Sigma} := (0, T) \times \partial\hat{\Omega}$; $\beta : R \rightarrow R$ is a nondecreasing function on R such that

$$(1.1) \quad \begin{cases} \beta(0) = 0, |\beta(r)| \geq C_0|r| - C'_0 & \text{for all } r \in R, \\ |\beta(r) - \beta(r')| \leq L_0|r - r'| & \text{for all } r, r' \in R, \end{cases}$$

where $C_0 > 0$, $C'_0 \geq 0$ and $L_0 > 0$ are constants. Here we suppose that $f \in L^2(\hat{Q})$, $g \in W^{2,2}(0, T; L^2(\hat{\Omega})) \cap L^2(0, T; H^2(\hat{\Omega}))$ and $u_0 \in L^2(\hat{\Omega})$. In this paper, u represents the enthalpy and $\beta(u)$ the temperature.

Now we give the weak formulation of $SP(\Omega)$.

DEFINITION 1.1. A function $u : [0, T] \rightarrow L^2(\Omega)$ is a weak solution of $SP(\Omega)$, if the following three conditions (w1) – (w3) are satisfied:

- (w1) $u \in C_w([0, T]; L^2(\Omega))$, $u(0) = u_0$;
- (w2) $\beta(u) \in L^2(0, T; H^1(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$;
- (w3) $-\int_{Q(\Omega)} u \eta_t dx dt + \int_{Q(\Omega)} \nabla \beta(u) \nabla \eta dx dt = \int_{Q(\Omega)} f \eta dx dt$
for all $\eta \in L^2(0, T; H_0^1(\Omega))$ with $\eta_t \in L^2(Q(\Omega))$ and $\eta(0, \cdot) = \eta(T, \cdot) = 0$.

REMARK 1.1. (1) In (w3) of Definition 1.1, it is enough to take as test function η

any smooth function of the form ρz , with $\rho \in \mathcal{D}(0, T)(= \{\rho \in C^\infty(R); \text{supp } \rho \subset (0, T)\})$ and $z \in H_0^1(\Omega)$.

(2) We denote by $C_w([0, T]; L^2(\Omega))$ the space of all weakly continuous functions from $[0, T]$ to $L^2(\Omega)$ and by $\langle \cdot, \cdot \rangle_\Omega$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now we introduce the notion of convergence of closed convex sets in a Banach space X , which is due to Mosco [13]. Let $\{K_n\}$ be a sequence of closed convex sets in X and K be a closed convex set in X . Then we say " $K_n \rightarrow K$ in X as $n \rightarrow \infty$ (in the sense of Mosco)" if the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in K_{n_k}$, and $z_k \rightarrow z$ weakly in X as $k \rightarrow \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset X$ such that

$$z_n \in K_n, n = 1, 2, \dots, \text{ and } z_n \rightarrow z \text{ in } X \text{ as } n \rightarrow \infty.$$

We denote by χ_Ω the characteristic function of Ω in $\hat{\Omega}$ for any subset Ω of $\hat{\Omega}$. We put

$$O := \{\Omega \subset \hat{\Omega}; \Omega \text{ is a smooth subdomain of } \hat{\Omega}\}$$

and for each $\Omega \in O$ denote by $V(\Omega)$ the set

$$\{z \in H_0^1(\hat{\Omega}); z = 0 \text{ a.e. on } \hat{\Omega} - \Omega\}.$$

Clearly $V(\Omega)$ is a closed linear subspace of $H_0^1(\hat{\Omega})$.

We consider the shape optimization problem for any non-empty subset O_c of O which is compact in the following sense:

$$(C) \left\{ \begin{array}{l} \text{For any sequence } \{\Omega_n\} \subset O_c \text{ there is a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ with } \Omega \in O_c \\ \text{such that } \chi_{\Omega_{n_k}} \rightarrow \chi_\Omega \text{ in } L^1(\hat{\Omega}) \text{ as } k \rightarrow \infty \text{ and } V(\Omega_{n_k}) \rightarrow V(\Omega) \text{ in } H_0^1(\hat{\Omega}) \\ \text{as } k \rightarrow \infty \text{ (in the sense of Mosco).} \end{array} \right.$$

We give below typical examples of O_c , which are very important in the application of our main results

EXAMPLE 1.1. (1) Let $\hat{\Omega}$ and O be the same as stated before. Let Θ be the class of

all C^1 -diffeomorphisms from $\overline{\hat{\Omega}}$ onto itself. Here we give Θ the topology induced from $C^1(\overline{\hat{\Omega}})$. Let Ω' be a smooth subdomain of $\hat{\Omega}$ with $\overline{\Omega'} \subset \hat{\Omega}$. For a given a non-empty compact subset Θ_c of Θ , we put

$$(1.2) \quad O_c = \{\theta(\Omega'); \theta \in \Theta_c\}.$$

Then this subset O_c of O satisfies condition (C).

Let $\{\Omega_n = \theta_n(\Omega')\}$ be any sequence in O_c . Then, by the compactness of Θ_c , there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \rightarrow \theta$ in $C^1(\overline{\hat{\Omega}})$ as $k \rightarrow \infty$ for some $\theta \in \Theta_c$. We see easily that $\chi_{\Omega_{n_k}} \rightarrow \chi_{\Omega}$, with $\Omega = \theta(\Omega')$, in $L^1(\hat{\Omega})$ as $k \rightarrow \infty$. Moreover, $V(\Omega_{n_k}) \rightarrow V(\Omega)$ in $H_0^1(\Omega)$ as $k \rightarrow \infty$ (in the sense of Mosco). In fact, if $z_{k'} \rightarrow z$ weakly in $H_0^1(\hat{\Omega})$ as $k' \rightarrow \infty$ for a subsequence $\{n_{k'}\}$ and $z_{k'} \in V(\Omega_{n_{k'}})$, then $\tilde{z}_{k'}(x) = z_{k'}(\theta_{n_{k'}} \circ \theta^{-1}(x)) \in V(\Omega)$ and $\tilde{z}_{k'} \rightarrow z(\theta \circ \theta^{-1}) = z$ weakly in $H_0^1(\hat{\Omega})$. So we see that $z \in V(\Omega)$. Also, let $z \in V(\Omega)$ and put $z_k(x) := z(\theta \circ \theta_{n_k}^{-1}(x)) \in V(\Omega_{n_k})$. Then, clearly, we have $z_k \rightarrow z$ in $H_0^1(\hat{\Omega})$.

EXAMPLE 1.2. Let $\hat{\Omega} := \{x; |x| < 2\} \subset \mathbb{R}^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \leq \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Here we put $O_c := \{\Omega_a; 0 < a \leq \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset O_c of O satisfies condition (C).

In fact, by [13; Lemma 1.8], the 2-capacity of any singleton is zero. Then, by [13], we see that $V(\Omega_a) \rightarrow V(\Omega)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a \rightarrow 0$. In the other hand, by the same argument as in Example 1.1, we obtain that $V(\Omega_{a'}) \rightarrow V(\Omega_a)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a' \rightarrow a$. Hence O_c satisfies condition (C). \diamond

In the case of Example 1.1, problems $SP(\Omega)$ can be reformulated as degenerate parabolic equations on the fixed domain Ω' by using the variable transformation $y = \theta^{-1}(x)$. However, in the case of Example 1.2, the situation is quite different, because there is no C^1 -diffeomorphism between domains Ω_a and Ω .

Based on an abstract result of [1] about the solvability of $SP(\Omega)$, we consider a shape optimization problem. For a given non-empty subset O_c of O , our optimization problem,

denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \quad \text{Find } \Omega_* \in O_c \text{ such that } J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

$$(1.3) \quad J(\Omega) = \frac{1}{2} \int_{Q(\Omega)} |\beta(u_\Omega) - \beta_d|^2 dxdt + \frac{1}{2} \int_{\hat{Q}-Q(\Omega)} |g|^2 dxdt \quad \text{for } \Omega \in O,$$

u_Ω is the weak solution of $SP(\Omega)$, and β_d is a given function in $L^2(\hat{Q})$.

In real problem, the driving variables are f, g and Ω . But, in this paper, we are interested in the effect of the domain Ω for the shape optimization. So, we fix the functions f and g , and take Ω as the driving variable.

The main results are stated in the following theorems. To prove the existence of solutions to $P(O_c)$, an important part is to show the continuous dependence of weak solution $u = u_\Omega$ to $SP(\Omega)$ upon $\Omega \in O$.

THEOREM 1.1. *Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \rightarrow V(\Omega)$ in $H_0^1(\hat{\Omega})$ as $n \rightarrow \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \rightarrow \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \rightarrow \infty$. Also, denote by u_n and u the weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \rightarrow \infty$,*

$$(1.4) \quad (u_n(t), z)_{\Omega_n} \rightarrow (u(t), z)_\Omega \quad \text{for any } z \in L^2(\hat{\Omega})$$

and

$$(1.5) \quad \tilde{\beta}(u_n) \rightarrow \tilde{\beta}(u) \quad \text{in } L^2(\hat{Q}).$$

Here we denote by $(\cdot, \cdot)_{\Omega'}$ the inner product in $L^2(\Omega')$ and put

$$\tilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \text{in } Q(\Omega'), \\ g & \text{in } \hat{Q} - Q(\Omega'), \end{cases}$$

for any $\Omega' \in O$.

The next theorem is concerned with the existence of a solution to $P(O_c)$.

THEOREM 1.2. *Problem $P(O_c)$ has at least one optimal solution Ω_* .*

We shall prove Theorems 1.1 and 1.2 in section 3.

2. Uniform estimates for the weak solutions to $SP(\Omega)$

In this section, we obtain some results from [1] on the existence, uniqueness and uniform estimates for weak solutions to $SP(\Omega)$. We use the following notations.

For simplicity, we denote by H the space $L^2(\hat{\Omega})$ and by X the Sobolev space $H_0^1(\hat{\Omega})$. Moreover, $|\cdot|_H$ stands for the norm in H and (\cdot, \cdot) the inner product in H . For each $\Omega \in O$, we define a bilinear form $a_\Omega(\cdot, \cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u, v) := \int_{\Omega} \nabla u \nabla v dx \quad \text{for all } u, v \in H^1(\Omega),$$

and denote by F_Ω the duality mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z \rangle := a_\Omega(v, z) \quad \text{for all } v, z \in H_0^1(\Omega),$$

where $\langle \cdot, \cdot \rangle_\Omega$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$. In particular, we put $a(\cdot, \cdot) := a_{\hat{\Omega}}(\cdot, \cdot)$.

According to the abstract result of [1; Theorem 2.1], problem $SP(\Omega)$ has a unique weak solution u such that $u \in W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$ for any $\Omega \in O$. In fact, the weak solution u is obtained as a unique solution of the following evolution problem in $H^{-1}(\Omega)$:

$$(2.1) \quad \begin{cases} u'(t) + F_\Omega(\beta(u(t)) - g(t)) = f(t) + \Delta g(t) & \text{for a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

We give some uniform estimates for weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 *There exists a positive constant M_1 independent of Ω such that*

$$(2.2) \quad \|u_\Omega\|_{L^\infty(0, T; L^2(\Omega))} \leq M_1, \quad \|\beta(u_\Omega)\|_{L^2(0, T; H^1(\Omega))} \leq M_1$$

$$(2.3) \quad \|t^{1/2} \frac{d}{dt} \beta(u_\Omega)\|_{L^2(0, T; L^2(\Omega))} \leq M_1, \quad \|t^{1/2} \beta(u_\Omega)\|_{L^\infty(0, T; H^1(\Omega))} \leq M_1$$

for all $\Omega \in O$, where u_Ω is the weak solution of $SP(\Omega)$.

Proof. As was seen in [1], problem $SP(\Omega)$ is able to be approximated by non-degenerated problem $SP(\Omega)^\varepsilon$, $\varepsilon \in (0, 1]$:

$$SP(\Omega)^\varepsilon \begin{cases} u_t - \Delta \beta^\varepsilon(u) = f & \text{in } Q(\Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ \beta^\varepsilon(u) = g & \text{on } \Sigma(\Omega), \end{cases}$$

where $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$, $r \in R$.

In fact, this problem has one and only one weak solution $u^\varepsilon \in C([0, T]; L^2(\Omega))$ such that $t^{1/2} \frac{d}{dt} \beta^\varepsilon(u^\varepsilon) \in L^2(Q(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \in L^2(0, T; H^1(\Omega))$. Moreover, we see that $u^\varepsilon \rightarrow u_\Omega$ in $C_w([0, T]; L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \rightarrow \beta(u_\Omega)$ weakly in $L^2(0, T; H^1(\Omega))$ as $\varepsilon \rightarrow 0$. After some calculations, we obtain that there is a positive constant C' independent of ε and Ω such that

$$(2.4) \quad \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla(\beta^\varepsilon(u^\varepsilon(t)))\|_{L^2(\Omega)}^2 dt \leq C'.$$

Moreover, multiply both sides of $u_t - \Delta \beta^\varepsilon(u^\varepsilon) = f$ by $t \frac{d}{dt}(\beta^\varepsilon(u^\varepsilon) - g)$ and integrate over $Q(\Omega)$. Then, by (2.4), we have

$$(2.5) \quad \begin{aligned} \|t^{1/2} \beta^\varepsilon(u^\varepsilon)\|_{L^\infty(0, T; H^1(\Omega))} &\leq C'', \quad \|t^{1/2} \frac{d}{dt} \beta^\varepsilon(u^\varepsilon)\|_{L^2(0, T; L^2(\Omega))} \leq C'', \\ &\text{for any } \varepsilon \in (0, 1] \text{ and } \Omega \in O, \end{aligned}$$

where C'' is a constant independent of $\varepsilon \in (0, 1]$ and $\Omega \in O$. Therefore, letting $\varepsilon \rightarrow 0$, we see that (2.2) and (2.3) hold. \diamond

3. Proofs of Theorems 1.1 and 1.2

First we prove Theorem 1.1.

Proof of THEOREM 1.1. Let consider the function $u_g \in L^\infty(0, T; H)$ such that $g(t, x) = \beta(u_g(t, x))$ in \hat{Q} . Here, we put

$$\tilde{u}_n = \begin{cases} u_n & \text{in } Q_n := Q(\Omega), \\ u_g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Then, we see that $\tilde{u}_n \in L^\infty(0, T; H)$. Moreover, we put $v_n := \beta(\tilde{u}_n)$ in \hat{Q} . By using Lemma 2.1, there exist a subsequence $\{n_k\}$ of $\{n\}$, $v \in L^2(0, T; H^1(\hat{\Omega}))$ and $\tilde{u} \in L^\infty(0, T; H)$ such that

$$(3.1) \quad \tilde{u}_{n_k} \rightarrow \tilde{u} \quad \text{weakly}^* \text{ in } L^\infty(0, T; H)$$

and

$$(3.2) \quad \begin{cases} v_{n_k} \rightarrow v & \text{weakly in } L^2(0, T; H^1(\hat{\Omega})), \\ v_{n_k}(t) \rightarrow v(t) & \text{weakly in } H^1(\hat{\Omega}) \text{ for all } t \in (0, T]. \end{cases}$$

By using Ascoli-Arzelà's theorem and Lemma 2.1, we easily verify that

$$v_{n_k} \rightarrow v \text{ in } L^2(0, T; H) \text{ as } k \rightarrow \infty.$$

Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in \hat{Q} , from (3.1) and (3.2) we show that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in (0, T]$.

Next, let z be any function in $V(\Omega)$ and ρ be any function in $\mathcal{D}(0, T)$. By the assumptions of Theorem 1.1, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \rightarrow z$ in X . Then by the definition of solution to $SP(\Omega)$ we have

$$-\int_0^T (u_{n_k}(t), z_{n_k})_{\Omega_{n_k}} \rho'(t) dt + \int_0^T a_{\Omega_{n_k}}(v_{n_k}(t), z_{n_k}) \rho(t) dt = \int_0^T (f(t), z_{n_k})_{\Omega_{n_k}} \rho(t) dt.$$

Letting $k \rightarrow \infty$, by $z_{n_k} = 0$ a.e. on $\hat{\Omega} - \Omega_{n_k}$ we obtain

$$-\int_0^T (\tilde{u}(t), z) \rho'(t) dt + \int_0^T a(v(t), z) \rho(t) dt = \int_0^T (f(t), z) \rho(t) dt.$$

This shows that $u = \tilde{u}|_{Q(\Omega)}$ is the solution of $SP(\Omega)$. \diamond

Proof of THEOREM 1.2. Since $J(\Omega) \geq 0$, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J(\Omega_n) \rightarrow J_* := \inf\{J(\Omega); \Omega \in O_c\}$$

Then, by the compactness of O_c , there are a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ and $\Omega_* \in O_c$ such that $V(\Omega_{n_k}) \rightarrow V(\Omega_*)$ in X (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_{n_k}} \rightarrow \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $k \rightarrow \infty$. Now, denote by u_k the weak solution of $SP(\Omega_{n_k})$ and by u_* the weak solution of $SP(\Omega_*)$. Then put

$$v_k := \begin{cases} \beta(u_k) & \text{in } Q_k = Q(\Omega_{n_k}), \\ g & \text{in } \hat{Q} - Q_k, \end{cases}$$

and

$$v := \begin{cases} \beta(u_*) & \text{in } Q = Q(\Omega_*), \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_k \rightarrow v$ in $L^2(0, T; H)$ as $k \rightarrow \infty$. Then we see that

$$J(\Omega_{n_k}) \rightarrow J(\Omega_*).$$

Therefore $J(\Omega_*) = J_*$. Hence Ω_* is a solution of $P(O_c)$. \diamond

4. Approximations for $SP(\Omega)$ and $P(O_c)$

In this section, from some numerical points of view, we discuss approximations of $SP(\Omega)$ and $P(O_c)$ by smooth problems. At first, we introduce the approximation β^ε and χ_Ω^ν for β and χ_Ω , respectively.

Let $\{\beta^\varepsilon\} = \{\beta^\varepsilon; 0 < \varepsilon \leq 1\}$ be a family of (smooth) functions $\beta^\varepsilon : R \rightarrow R$ such that

$$(\beta) \begin{cases} |\beta^\varepsilon(r) - \beta(r)| \leq \varepsilon(|r| + 1) & \text{for all } r \in R; \\ \beta^\varepsilon(0) = 0, |\beta^\varepsilon(r) - \beta^\varepsilon(r')| \leq \tilde{L}_0 |r - r'| & \text{for all } r, r' \in R, \\ \frac{d}{dr} \beta^\varepsilon(r) \geq \varepsilon & \text{for a.e. } r \in R, \end{cases}$$

where $\tilde{L}_0 > 0$ is a constant independent of ε .

Next, let $\{\chi_\Omega^\nu\} = \{\chi_\Omega^\nu; 0 < \nu \leq 1, \Omega \in O_c\}$ be a family of smooth functions on $\hat{\Omega}$ and suppose that the following two conditions ($\chi 1$) and ($\chi 2$) hold :

$$(\chi 1) \quad 0 \leq \chi_\Omega \leq \chi_\Omega^\nu \leq 1 \text{ in } \hat{\Omega} \text{ and } \text{supp } (\chi_\Omega^\nu) \subset \{x \in \hat{\Omega}; \text{dist}(x, \Omega) \leq \nu\}$$

for any $\nu \in (0, 1]$ and $\Omega \in O_c$.

$$(\chi 2) \quad \text{For each } \nu \in (0, 1], \{\chi_\Omega^\nu; \Omega \in O_c\} \text{ is compact in } L^1(\hat{\Omega}).$$

We give below typical examples of approximations β^ε and χ_Ω^ν for β and χ_Ω , respectively, which satisfy the conditions mentioned above.

EXAMPLE 4.1. (1) We define $\beta^\varepsilon : R \rightarrow R$ by $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$ for any $r \in R$. Then, the family of $\{\beta^\varepsilon\}$ satisfies the condition (β) for $\tilde{L}_0 = L_0 + 1$ where L_0 is the constant of (1.1).

(2) Let $\hat{\Omega}$, Ω' and O_c be the same as in Example 1.1. Now, for each $\nu \in (0, 1]$ and $\Omega \in O_c$, we denote by $\Omega(\frac{\nu}{2})$ the set $\{x \in \hat{\Omega}; \text{dist}(x, \Omega) \leq \frac{\nu}{2}\}$. Let χ_Ω^ν be the regularization of $\chi_{\Omega(\frac{\nu}{2})}$ by means of usual mollifiers on $\hat{\Omega}$. Clearly, we see that $(\chi 1)$ holds. Also, we obtain that $(\chi 2)$ holds. Because we can prove that

$$(4.1) \quad \text{if } \Omega_n = \theta_n(\Omega'), \theta_n \rightarrow \theta \text{ in } C^1(\bar{\hat{\Omega}}) \text{ and } \Omega = \theta(\Omega'), \text{ then } \chi_{\Omega_n} \rightarrow \chi_\Omega \text{ in } L^1(\hat{\Omega}).$$

Now, we define the approximate problem $SP(\Omega)^{\varepsilon\nu\mu}$, $\varepsilon, \nu, \mu \in (0, 1]$, by using the penalty method for $SP(\Omega)$:

$$SP(\Omega)^{\varepsilon\nu\mu} \begin{cases} u_t - \Delta \beta^\varepsilon(u) = f - \frac{1 - \chi_\Omega^\nu}{\mu} (\beta^\varepsilon(u) - g) & \text{in } \hat{Q}, \\ u(0, \cdot) = u_0 & \text{in } \hat{\Omega}, \\ \beta^\varepsilon(u) = g & \text{on } \hat{\Sigma}. \end{cases}$$

Here we give the weak formulation of $SP(\Omega)^{\varepsilon\nu\mu}$.

DEFINITION 4.1. A function $u : [0, T] \rightarrow H$ is a solution of $SP(\Omega)^{\varepsilon\nu\mu}$, if the following three conditions (aw1) – (aw3) are satisfied:

$$(aw1) \quad u \in C([0, T]; H) \cap W_{loc}^{1,2}((0, T]; H) \cap L^2(0, T; H^1(\hat{\Omega})), \quad u(0) = u_0 \text{ in } \hat{\Omega};$$

$$(aw2) \quad \beta^\varepsilon(u(t)) - g(t) \in X \text{ for a.e. } t \in [0, T];$$

$$(aw3) \quad \langle u'(t), z \rangle_{\hat{\Omega}} + a(\beta^\varepsilon(u(t)), z) = (f(t) - \frac{1 - \chi_\Omega^\nu}{\mu} (\beta^\varepsilon(u(t)) - g(t)), z) \\ \text{for any } z \in X, \text{ a.e. } t \in [0, T].$$

According to the abstract result in [9; Chapter 2] (or [10]), problem $SP(\Omega)^{\varepsilon\nu\mu}$ has a unique solution u .

Our approximate optimization problem $P(O_c)^{\varepsilon\nu\mu}$, associated with $SP(\Omega)^{\varepsilon\nu\mu}$, is formu-

lated as follows:

$$P(O_c)^{\varepsilon\nu\mu} \quad \text{Find } \Omega_*^{\varepsilon\nu\mu} \in O_c \text{ such that } J^{\varepsilon\nu\mu}(\Omega_*^{\varepsilon\nu\mu}) = \inf_{\Omega \in O_c} J^{\varepsilon\nu\mu}(\Omega),$$

where

$$J^{\varepsilon\nu\mu}(\Omega) = \frac{1}{2} \int_{\hat{Q}} \chi_{\Omega}^{\nu} |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}) - \beta_d|^2 dxdt + \frac{1}{2} \int_{\hat{Q}} (1 - \chi_{\Omega}^{\nu}) |g|^2 dxdt,$$

$u_{\Omega}^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Next, we give the convergence results in the following theorem.

THEOREM 4.1. *We have the following statements (1) and (2):*

(1) *For each $\varepsilon, \nu, \mu \in (0, 1]$, $P(O_c)^{\varepsilon\nu\mu}$ has at least one solution.*

(2) *Let $\{\varepsilon_n\}, \{\nu_n\}, \{\mu_n\}$ be null sequences and let $\{\Omega_n\} \subset O_c$ and $\Omega \in O_c$ such that $V(\Omega_n) \rightarrow V(\Omega)$ in X as $n \rightarrow \infty$ (in the sense of Mosco), $\chi_{\Omega_n}^{\nu_n} \rightarrow \chi_{\Omega}$ in $L^1(\hat{\Omega})$ as $n \rightarrow \infty$. Denote by u_n the solution of $SP(\Omega_n)^{\varepsilon_n\nu_n\mu_n}$. Then as $n \rightarrow \infty$,*

$$\begin{cases} \chi_{\Omega_n} u_n \rightarrow \chi_{\Omega} u & \text{weakly* in } L^{\infty}(0, T; H), \\ \beta^{\varepsilon_n}(u_n) \rightarrow v & \text{in } L^2(0, T; H) \text{ and weakly in } L^2(0, T; H^1(\hat{\Omega})), \end{cases}$$

Moreover u is the weak solution of $SP(\Omega)$ and

$$v = \begin{cases} \beta(u) & \text{in } Q = (0, T) \times \Omega, \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

In particular, if Ω_n is a solution of $P(O_c)^{\varepsilon\nu\mu}$ with $\varepsilon = \varepsilon_n, \nu = \nu_n$ and $\mu = \mu_n$ for $n = 1, 2, \dots$, then Ω is a solution of $P(O_c)$.

In this theorem, $\{\varepsilon_n\}$, $\{\nu_n\}$, and $\{\mu_n\}$ are chosen independently. This is very convenient for numerical computation. Moreover, we show that $P(O_c)^{\varepsilon\nu\mu}$ converges to $P(c)$ in some sense.

5. Energy estimates for $SP(\Omega)^{\varepsilon\nu\mu}$

For the proof of Theorem 4.1, we prepare some lemmas on energy estimates for solutions of $SP(\Omega)^{\varepsilon\nu\mu}$ with respect to $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$.

LEMMA 5.1. *There is a positive constant M_2 such that*

$$(5.1) \quad |u_{\Omega}^{\varepsilon\nu\mu}|_{L^{\infty}(0,T;H)} \leq M_2, |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu})|_{L^2(0,T;H^1(\widehat{\Omega}))} \leq M_2$$

and

$$(5.2) \quad \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi_{\Omega}^{\nu}) |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}) - g|^2 dx dt \leq M_2$$

for all $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$, where $u_{\Omega}^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. For $0 < \nu, \mu \leq 1, \Omega \in O, 0 \leq t \leq T$, we introduce a proper lower semi-continuous convex function $\varphi_{\Omega}^{\nu\mu}$ on H as follows:

$$(5.3) \quad \varphi_{\Omega}^{\nu\mu}(t, z) = \begin{cases} \frac{1}{2} |\nabla z|_H^2 + \frac{1}{2\mu} \int_{\widehat{\Omega}} (1 - \chi_{\Omega}^{\nu}) |z - g(t)|^2 dx & \text{for } z - g(t) \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

We easily see that the subdifferential $\partial\varphi_{\Omega}^{\nu\mu}(t, \cdot)$ in H is singlevalued in H and

$$(5.4) \quad z^* = \partial\varphi_{\Omega}^{\nu\mu}(t, z) \Leftrightarrow \begin{cases} z - g(t) \in X, z^* \in H, \\ z^* = -\Delta z + \frac{1 - \chi_{\Omega}^{\nu}}{\mu} (z - g(t)) \in H. \end{cases}$$

By using (5.4), we can show that $SP(\Omega)^{\varepsilon\nu\mu}$ can be reformulated by the following evolution problem in H :

$$(5.5) \quad \begin{cases} u'(t) + \partial\varphi_{\Omega}^{\nu\mu}(t, \beta^{\varepsilon}(u(t))) = f(t) & \text{in } H \text{ for a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

For simplicity, we write u for $u_{\Omega}^{\varepsilon\nu\mu}$, χ for χ_{Ω}^{ν} and $\varphi(t, \cdot)$ for $\varphi_{\Omega}^{\nu\mu}(t, \cdot)$. Multiplying $u'(t) + \partial\varphi(t, \beta^{\varepsilon}(u(t))) = f(t)$ by $\beta^{\varepsilon}(u(t)) - g(t)$, by using (5.4), we obtain

$$\begin{aligned} & (u'(t), \beta^{\varepsilon}(u(t)) - g(t)) + a(\beta^{\varepsilon}(u(t)), \beta^{\varepsilon}(u(t)) - g(t)) \\ & + \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi) |\beta^{\varepsilon}(u(t)) - g(t)|^2 dx \\ & = (f(t), \beta^{\varepsilon}(u(t)) - g(t)). \end{aligned}$$

After some calculations, we obtain the following inequality:

$$(5.6) \quad \begin{aligned} & \frac{d}{dt} \left\{ \int_{\widehat{\Omega}} \widehat{\beta^{\varepsilon}}(u(t)) dx - (g(t), u(t)) \right\} \\ & + R_1 \left\{ |\nabla(\beta^{\varepsilon}(u(t)) - g(t))|_H^2 + \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi) |\beta^{\varepsilon}(u(t)) - g(t)|^2 dx \right\} \\ & \leq R_2 \left\{ \int_{\widehat{\Omega}} \widehat{\beta^{\varepsilon}}(u(t)) dt - (g(t), u(t)) \right\} \\ & + R_3 (1 + |g(t)|_{H^1(\widehat{\Omega})}^2 + |g'(t)|_H^2 + |f(t)|_H^2) \end{aligned}$$

where $R_i, i = 1, 2, 3$, are positive constants independent of ε, ν, μ and Ω . By using Gronwall's inequality and (5.6), we show (5.1) and (5.2) for a positive constant M_2 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. \diamond

LEMMA 5.2. *There is a positive constant M_3 such that*

$$(5.7) \quad |t^{1/2} \beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})|_{L^\infty(0,T;H^1(\hat{\Omega}))} \leq M_3, |t^{1/2} \frac{d}{dt} \beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})|_{L^2(0,T;H)} \leq M_3,$$

and

$$(5.8) \quad \sup_{t \in (0,T]} \frac{t}{\mu} \int_{\hat{\Omega}} (1 - \chi_\Omega^\nu) |\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu}(t)) - g(t)|^2 dx \leq M_3,$$

for all $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$, where $u_\Omega^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. Simply write u for $u_\Omega^{\varepsilon\nu\mu}$ and $\tilde{\beta}$ for $\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})$. Let us consider the convex function $\psi := \psi_\Omega^{\nu\mu}$ on H given by

$$\psi_\Omega^{\nu\mu}(z) = \begin{cases} \frac{1}{2} \|\nabla z\|_H^2 + \frac{1}{2\mu} \int_{\hat{\Omega}} (1 - \chi_\Omega^\nu) |z|^2 dx & \text{for } z \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, it is easy to see that ψ is proper lower semicontinuous and convex on H , and the subdifferential $\partial\psi$ is singlevalued in H . Besides,

$$z^* = \partial\psi(z) \Leftrightarrow \begin{cases} z \in X, z^* \in H, \\ z^* = -\Delta z + \frac{1 - \chi_\Omega^\nu}{\mu} z \in H. \end{cases}$$

Moreover, by the standard argument of convex analysis, we have

$$(5.9) \quad \frac{d}{dt} \psi(z(t)) = (\partial\psi(z(t)), z'(t)) \text{ for } z \in W^{1,2}(0, T; H).$$

Then, by using (5.4) and (5.5), we see that

$$\begin{aligned} & (u'(t), \tilde{\beta}'(t) - g'(t)) + (-\Delta(\tilde{\beta}(t) - g(t)) + \frac{1 - \chi_\Omega^\nu}{\mu}(\tilde{\beta}(t) - g(t)), \tilde{\beta}'(t) - g'(t)) \\ &= (f(t) + \Delta g(t), \tilde{\beta}'(t) - g'(t)). \end{aligned}$$

Then, by (5.9), we show that

$$\begin{aligned}
 (5.10) \quad & \frac{t}{2\tilde{L}_0} |\tilde{\beta}'(t)|_H^2 \\
 & + \frac{d}{dt} \left\{ \frac{t}{2} |\nabla(\tilde{\beta}(t) - g(t))|_H^2 - t(u(t), g'(t)) + \frac{t}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^{\nu}) |\tilde{\beta}(t) - g(t)|^2 dx \right\} \\
 & \leq T |f(t) + \Delta g(t)|_H \{ |g'(t)|_H + \frac{\tilde{L}_0}{2} |f(t) + \Delta g(t)|_H \} + T |u(t)|_H \cdot |g''(t)|_H \\
 & + \frac{1}{2} |\nabla(\tilde{\beta}(t) - g(t))|_H^2 - (u(t), g'(t)) + \frac{1}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^{\nu}) |\tilde{\beta}(t) - g(t)|^2 dx.
 \end{aligned}$$

Here, integrating (5.10) over $[0, t]$ and using Lemma 5.1, we derive the estimates (5.7) and (5.8) for some positive constant M_3 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. \diamond

6. Proof of Theorem 4.1.

Now we prove Theorem 4.1.

Proof of (1) of THEOREM 4.1. Fix $\varepsilon, \nu, \mu \in (0, 1]$ and put $I_* = \inf\{J^{\varepsilon\nu\mu}(\Omega); \Omega \in O_c\} \geq 0$. Then, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J^{\varepsilon\nu\mu}(\Omega_n) \rightarrow I_* \text{ (as } n \rightarrow \infty \text{)}.$$

By (χ_2) , there is a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ such that $V(\Omega_{n_k}) \rightarrow V(\Omega)$ in X (in the sense of Mosco) and $\chi_k := \chi_{\Omega_{n_k}}^{\nu} \rightarrow \chi_{\Omega}^{\nu} =: \chi$ in $L^1(\hat{\Omega})$ for some $\Omega \in O_c$. In a similar way to that of the proof of Theorem 1.1, we can prove that the solution $u_k := u_{\Omega_{n_k}}^{\varepsilon\nu\mu}$ converges to the weak solution $u := u_{\Omega}^{\varepsilon\nu\mu}$ of $SP(\Omega)^{\varepsilon\nu\mu}$ in the sense that

$$\begin{cases} u_k \rightarrow u & \text{in } L^2(0, T; H) \\ \beta^{\varepsilon}(u_k) \rightarrow \beta^{\varepsilon}(u) & \text{in } L^2(0, T; H) \end{cases}$$

Therefore

$$I_* = \lim_{k \rightarrow \infty} J^{\varepsilon\nu\mu}(\Omega_k) = J^{\varepsilon\nu\mu}(\Omega),$$

and we see that Ω is a solution of $P(O_c)^{\varepsilon\nu\mu}$. \diamond

Proof of (2) of Theorem 4.1. By Lemma 5.1 and Lemma 5.2, we may assume that

$$(6.1) \quad u_n \rightarrow \tilde{u} \text{ weakly* in } L^{\infty}([0, T]; H),$$

and

$$(6.2) \quad \begin{cases} \tilde{\beta}_n := \beta^{\varepsilon_n}(u_n) \rightarrow \beta(\tilde{u}) =: \tilde{\beta} \text{ in } C_{loc}((0, T]; H) \text{ and weakly in } L^2(0, T; H^1(\hat{\Omega})), \\ \tilde{\beta}_n(t) \rightarrow \tilde{\beta}(t) \text{ weakly in } H^1(\hat{\Omega}) \text{ for any } t \in (0, T]. \end{cases}$$

In fact, (6.1) and (6.2) are obtained in a similar way to the proof of Theorem 1.2. Moreover, by using (5.8) of Lemma 5.2 and (6.2), we have

$$\begin{cases} \chi_{\Omega_n} u_n \rightarrow \chi_{\Omega} u \text{ weakly* in } L^\infty(0, T; H), \\ \tilde{\beta}_n \rightarrow \tilde{\beta} \text{ in } L^2(0, T; H), \\ \int_{\hat{\Omega}} (1 - \chi_{\Omega_n}^{\nu_n}) |\tilde{\beta}_n(t) - g(t)|^2 dx \rightarrow 0 = \int_{\hat{\Omega}} (1 - \chi_{\Omega}) |\tilde{\beta}(t) - g(t)|^2 dx \\ \text{for any } t \in (0, T], \end{cases}$$

so that

$$(6.3) \quad \tilde{\beta}(t) - g(t) \in V(\Omega) \quad \text{for any } t \in (0, T].$$

Next, let ρ be any function in $\mathcal{D}(0, T)$. By assumption, for any $z \in V(\Omega)$, there is a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \rightarrow z$ in X . From (5.5) it follows that

$$\begin{aligned} & - \int_0^T (u_n(t), z_n) \rho(t) dt + \int_0^T a(\tilde{\beta}_n(t), z_n) \rho(t) dt + \frac{1}{\mu_n} \int_0^T ((1 - \chi_{\Omega_n}^{\nu_n})(\tilde{\beta}_n - g)(t), z_n) \rho(t) dt \\ & = \int_0^T (f(t), z_n) \rho(t) dt. \end{aligned}$$

Since $(1 - \chi_{\Omega_n}^{\nu_n})z_n = 0$ a.e. on $\hat{\Omega}$, as $n \rightarrow \infty$, we get that

$$\int_0^T \langle \tilde{u}'(t), z \rho(t) \rangle_{\hat{\Omega}} dt + \int_0^T a(\tilde{\beta}(t), z) \rho(t) dt = \int_0^T (f(t), z) \rho(t) dt.$$

Therefore \tilde{u} is the weak solution of $SP(\Omega)$.

In particular, let Ω_n be a solution of $P(O_c)^{\varepsilon_n \nu_n \mu_n}$ for each n . Just as above

$$J^{\varepsilon_n \nu_n \mu_n}(\Omega_n) \rightarrow J(\Omega)$$

and

$$J^{\varepsilon_n \nu_n \mu_n}(\Omega') \rightarrow J(\Omega') \quad \text{for any } \Omega' \in O_c.$$

Therefore, for any $\Omega' \in O_c$,

$$J(\Omega') = \lim_{n \rightarrow \infty} J^{\varepsilon_n \nu_n \mu_n}(\Omega') \geq \lim_{n \rightarrow \infty} J^{\varepsilon_n \nu_n \mu_n}(\Omega_n) = J(\Omega).$$

This shows that Ω is a solution of $P(O_c)$. \diamond

For the detailed proofs of all results stated in this note, see the forthcoming paper [17].

References

- [1] A.Damlamian, Some results on the multi-phase Stefan problem, Comm. P.D.E. 2. (1977), 1017 - 1044.
- [2] A.Damlamian and N.Kenmochi, Asymptotic behavior of solutions to a multi-phase Stefan problems. Japan J. Appl. Math., 3(1986), 15 - 36.
- [3] A.Freidman, The Stefan problem in several space variables, Trans. Amer. Math. Soc. 133(1968), 51 - 87.
- [4] N.Fujii, Existence of an optimal domain optimization problem, Lecture Notes in Control and Information Sciences 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 251 - 258.
- [5] Y.Goto, N.Fujii and Y.Muramatsu, Second Order Necessary Optimality Conditions for Domain Optimization Problem with a Neumann Problem, Lecture Notes in Control and Information Sciences 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 259 - 268.
- [6] J.Haslinger and P.Neittaanmäki, Finite Element Approximation for Optimal Shape Design: Theory and Applications, John Willy & Sons Ltd, Chichester-New York-Brisbane-Toronto-Singapore, 1988.
- [7] J.Haslinger, P.Neittaanmäki and D.Tiba, On state constrained optimal shape design problems, Optimal Control of Partial Differential Equations II, ISNM 78, Birkhäuser, Basel-Boston-Stuttgart, 1987, 109 - 122.

- [8] S.L. Kamenomostskaja, On Stefan's problem, *Mat. Sb.* 53(1961), 489 - 514.
- [9] N.Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, *Bull. Fac. Education, Chiba Univ.* 30(1981), 1 - 87.
- [10] N.Kenmochi and I.Pawlow, A class of nonlinear elliptic-parabolic equations with time-dependent constraints, *Nonlinear Anal.* 10(1986), 1181 - 1202.
- [11] M.Koda, Sensitivity Analysis of A Descriptor Distributed Parameter System and Its Application to Shape Optimization, *Lecture Notes in Control and Information Sciences* 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 241 - 250.
- [12] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr. 23, Amer. Math. Soc., Providence R.I., 1968.
- [13] U.Mosco, Convergence of convex sets and of solutions of variational inequalities, *Advances Math.* 3(1969), 510 - 585.
- [14] P.Neittaanmäki, J.Sokolowski and J.P.Zolesio, *Optimization of the Domain in Elliptic Variational Inequalities*, *App. Math. Optim.* 18: 58 - 98 (1988) Springer.
- [15] I.Pawlow, Analysis and control in free boundary problems, *Bull. Fac. Education, Chiba Univ.* 36(1988), 9 - 67.
- [16] C.Saguez, *Contrôle Optimal de Systemes a frontière libre*, Thèse, Univ. Technologique, Compiègne, 1980.
- [17] A.Kadoya and N.Kenmochi, Optimal Shape Design in Multi-Phase Stefan Problems, *Tech. Report Math. Sci.*, Vol.6, No.8, Chiba Univ., 1990.